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# Construction of $SU(3)$ irreps in canonical $SO(3)$ -coupled bases

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## Abstract

Alternative methods for defining canonical  $SO(3)$ -coupled bases for  $SU(3)$  irreps are considered and compared. It is shown that a basis that diagonalizes a particular linear combination of  $SO(3)$  invariants in the  $SU(3)$  universal enveloping algebra gives basis states that have good  $K$  quantum numbers in the asymptotic rotor-model limit.

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## 1. Introduction

A common problem in the construction of group or Lie algebra representations is to define a canonical basis in situations where multiplicities occur. For example, bases which reduce the subgroup chain

$$\begin{array}{ccccccc} SU(3) & \supset & SO(3) & \supset & SO(2) & & \\ (\lambda\mu) & & K & & L & & M \end{array} \quad (1)$$

are indexed by the quantum numbers  $(\lambda\mu)$ ,  $L$  and  $M$  of the respective groups  $SU(3)$ ,  $SO(3)$  and  $SO(2)$ . However, an extra label  $K$  is required to distinguish different irreps (irreducible representations) of  $SO(3)$  that occur in a given  $SU(3)$  irrep. This paper is concerned with useful ways to define orthogonal sets of such  $SO(3)$  irreps.

In principle, multiple occurrences of subgroup irreps can be defined in any arbitrary way. However, it is useful to have a well-defined ‘canonical’ definition that can be reproduced by anyone so that results derived by one person are meaningful to someone else. For example, in applications of group representations, considerable use is made of Clebsch–Gordan coupling and Racah recoupling coefficients that are defined for particular resolutions of the multiplicities that occur. Two kinds of multiplicity arise: one is the multiplicity in the choice of basis for each irrep. Another, of equal importance, is the multiplicity of different irreps that occur in

the decomposition of tensor products of irreps. In this paper, we address the resolution of the first of these two multiplicities.

The resolution of the multiplicity problems for  $SU(3)$  is of particular importance in nuclear physics where  $SU(3)$  representations are used to explain the origin of rotational bands in terms of the nuclear shell model. The interesting fact is that the  $SO(3)$  multiplicity is simply resolved, in the rotor model, by defining the multiplicity label  $K$  to be the component of the angular momentum  $L$  relative to a so-called *intrinsic axis* fixed in the body of the rotor. Thus, in using  $SU(3)$  to describe rotational bands in nuclei, it is desirable to choose basis states which correspond to the standard rotor-model choice in the limit in which the  $SU(3)$  irreps contract to those of the rotor model. How to make such a choice is not obvious because intrinsic axes are not defined for an  $SU(3)$  irrep. However, we shall show that, in addition to reducing the  $SO(3) \subset SO(2)$  groups, the standard basis states of a rotor, labeled by  $K$ ,  $L$  and  $M$  are eigenstates of an  $SO(3)$ -invariant operator in the universal enveloping algebra of the rotor model. Moreover, a parallel  $SO(3)$ -invariant operator exists in the  $SU(3)$  universal enveloping algebra. Thus, we derive the sought after resolution of the  $SU(3)$  multiplicity that does correspond, in a contraction limit, to the standard choice of the rotor model.

## 2. $SU(3)$ irreps and their asymptotic limits

The  $su(3)$  Lie algebra is spanned by five components of a quadrupole tensor  $\mathcal{Q}$  and three components of an angular momentum  $\mathbf{L}$ . (Note that we use upper-case letters to denote a Lie group, e.g.,  $SU(3)$ , and lower-case letters, e.g.,  $su(3)$ , for its Lie algebra.) As discussed in more detail below,  $su(3)$  has an İnönü–Wigner [1] contraction to the Lie algebra,  $rot(3)$ , of a rigid-rotor model. The latter Lie algebra is likewise spanned by five components of a quadrupole tensor  $\mathcal{Q}$  and three components of an angular momentum  $\mathbf{L}$ . Both  $su(3)$  and  $rot(3)$  have commutation relations

$$[L_k, L_{k'}] = -\sqrt{2}(1k, 1k'|1k+k')L_{k+k'}, \quad (2)$$

$$[L_k, \mathcal{Q}_v] = -\sqrt{6}(1k, 2v|2v+k)\mathcal{Q}_{v+k}. \quad (3)$$

However, they differ in the commutators of their  $\{\mathcal{Q}_v\}$  operators:

$$[\mathcal{Q}_v, \mathcal{Q}_\mu] = 3\sqrt{10}(2\mu, 2v|1\mu+v)L_{\mu+v} \times \begin{cases} 0 & \text{for } rot(3), \\ 1 & \text{for } su(3). \end{cases} \quad (4)$$

Thus, whereas  $su(3)$  is semi-simple, its contraction,  $rot(3)$ , is a semi-direct sum of an Abelian subalgebra, isomorphic to  $\mathbb{R}^5$ , and an  $so(3)$  angular-momentum algebra; we denote this by writing  $rot(3) \simeq [\mathbb{R}^5]so(3)$ . As shown in the following section, the conditions under which the  $so(3) \rightarrow rot(3)$  contraction apply are for states of finite angular momentum and for asymptotically large-dimensional  $su(3)$  irreps. When these conditions are satisfied, the matrix elements of the  $su(3)$  algebra approach those of the rotor model.

It has been shown in [2] that basis states for a generic  $(\lambda, \mu)$ , irrep of  $su(3)$ , are labeled by angular-momentum quantum numbers,  $L$  and  $M$ , with  $L$  running over the values

$$L = \begin{cases} \lambda + K, \lambda + K - 1, \dots, K & \text{for } K \neq 0 \\ \lambda, \lambda - 2, \dots, 0 \text{ or } 1 & \text{for } K = 0 \end{cases} \quad (5)$$

with

$$K = \mu, \mu - 2, \dots, 0 \text{ or } 1. \quad (6)$$

Thus, in the generic case, there is a multiplicity of states with given values of  $L$  and  $M$ , which can be indexed by  $K$  or any other convenient label.

There are several ways to construct  $SU(3)$  irreps in an  $SO(3)$ -coupled basis and derive the corresponding matrices representing elements of the  $su(3)$  Lie algebra. Irreps of the type  $(\lambda, 0)$  are particularly simple. They have orthonormal  $SO(3)$ -coupled bases given, without multiplicity, by a set of states

$$\{|LM\rangle; M = -L, \dots, +L, L = \lambda, \lambda - 2, \dots, 0 \text{ or } 1\}, \quad (7)$$

in which  $L$  runs over even or odd integer values according as  $\lambda$  is, respectively, even or odd. Reduced matrix elements for such multiplicity-free irreps have analytical expressions given [3, 4] (in natural units) by the equations

$$\langle L \| Q \| L \rangle = \sqrt{2L+1} \langle L0, 20 | L0 \rangle (2\lambda + 3), \quad (8)$$

$$\langle L+2 \| Q \| L \rangle = \sqrt{2L+1} \langle L0, 20 | L+2, 0 \rangle [4(\lambda - L)(\lambda + L + 3)]^{\frac{1}{2}}. \quad (9)$$

A systematic way to derive matrix elements for a generic  $SU(3)$  irrep was given [3, 4] in terms of vector coherent state [5, 6] theory. VCS methods were also used in a derivation of  $SU(3)$  Clebsch–Gordan coefficients in an  $SO(3)$ -coupled basis [7, 8]. Conversely, a set of  $SU(3)$  Clebsch–Gordan coefficients computed in an  $SO(3)$ -coupled basis enables one to derive the  $SO(3)$ -reduced matrices of the  $SU(3)$  quadrupole tensor in that basis. Examples of reduced matrix elements derived in this way are given below. Such methods do not give analytical expressions for generic irreps, for which there are multiplicities, However, analytical expressions are obtained [3, 4] in the asymptotic limits which are approached as either  $\lambda$  or  $\mu \rightarrow \infty$ .

In the following, we restrict consideration to  $su(3)$  irreps  $\{(\lambda, \mu)\}$  with  $\lambda \geq \mu$ . This is because of the well-known fact (as shown, for example, in [4]) that the irreps  $(\lambda, \mu)$  and  $(\mu, \lambda)$  are simply related. Specifically, if  $\Gamma_v^{(\lambda, \mu)}$  denotes the matrix representing the quadrupole operator  $Q_v$  in the  $su(3)$  irrep  $(\lambda, \mu)$ , then

$$\Gamma_v^{(\lambda, \mu)} = -\Gamma_v^{(\mu, \lambda)}. \quad (10)$$

With this restriction, asymptotic expressions for the  $su(3)$  quadrupole matrix elements are given for  $\lambda \rightarrow \infty$  by

$$\langle KL \| Q \| KL \rangle \sim \sqrt{2L+1} \langle LK, 20 | LK \rangle (\Lambda + \delta_{K,1} \sigma_{LL}) \quad (11)$$

$$\begin{aligned} \langle KL+1 \| Q \| KL \rangle &\sim \sqrt{2L+1} \langle LK, 20 | L+1, K \rangle \\ &\times \sqrt{(\Lambda - L - 1 + \delta_{K,1} \sigma_{L+1,L})(\Lambda + L + 1 + \delta_{K,1} \sigma_{L+1,L})} \end{aligned} \quad (12)$$

$$\begin{aligned} \langle KL+2 \| Q \| KL \rangle &\sim \sqrt{2L+1} \langle LK, 20 | L+2, K \rangle \\ &\times \sqrt{(\Lambda - 2L - 3 + \delta_{K,1} \sigma_{L+2,L})(\Lambda + 2L + 3 + \delta_{K,1} \sigma_{L+2,L})} \end{aligned} \quad (13)$$

$$\begin{aligned} \langle K+2, L' \| Q \| KL \rangle &= (-1)^{L'-L} \langle KL \| Q \| K+2, L' \rangle \\ &\sim \sqrt{(2L+1)(1 + \delta_{K,0})} \langle LK, 22 | L', K+2 \rangle \sqrt{\frac{3}{2}(\mu - K)(\mu + K + 2)}, \end{aligned} \quad (14)$$

where  $\Lambda = 2\lambda + \mu + 3$  and

$$\sigma_{L'L} = \frac{1}{2}(\mu + 1)(-1)^{\lambda+L} \times \begin{cases} -\frac{3L(L+1)}{3-L(L+1)} & \text{for } L' = L \\ L+1 & \text{for } L' = L+1 \\ -L & \text{for } L' = L-1 \\ -1 & \text{for } L' = L \pm 2. \end{cases} \quad (15)$$

**Table 1.** Comparison of quadrupole reduced matrix elements  $\langle K_f L_f \| Q \| K_i L_i \rangle$  for bases defined by diagonalizing the operator  $Q^{(1)} \cdot Q^{(2)}$  and by the I, II and III alternatives, as defined in the text, for the  $SU(3)$  irrep (32, 5). Values given by the asymptotic approximations of equations (11)–(15) are shown in the column headed A.S. Values for  $rot(3)$ , given by equations (16) and (17), are shown in the column headed  $ROT(3)$ .

$K_i L_i$	$K_f L_f$	$Q^{(1)} \cdot Q^{(2)}$	I	II	III	A.S.	$ROT(3)$
1; 3	1; 1	58.030 319	81.421 678	81.979 149	81.974 076	81.975 606	81.610 661
3; 3	1; 1	59.131 052	15.313 707	11.975 740	12.010 416	12	10.606 602
1; 3	1; 2	-71.091 833	-81.321 756	-80.940 834	-80.945 466	-80.944 425	-79.5
3; 3	1; 2	-40.223 759	7.666 261	10.980 970	10.946 736	10.954 451	9.682 458
1; 3	1; 3	-1.248 765	-61.784 810	-61.476 192	-61.482 530	-61.481 705	-63.531 095
3; 3	1; 3	-86.819 713	0	7.551 035	7.473 012	7.483 315	6.614 378
3; 3	3; 3	62.730 470	123.266 515	122.957 882	122.964 235	122.963 409	122.963 4092
1; 4	1; 2	90.971 036	99.475 476	100.490 358	100.480 593	100.484 540	101.737 583
3; 4	1; 2	44.061 450	17.991 295	10.900 948	10.990 508	10.954 451	9.682 458
1; 4	1; 3	-66.087 883	-52.793 962	-51.839 707	-51.849 622	-51.845 926	-51.693 575
3; 4	1; 3	69.315 355	14.450 569	12.953 794	12.961 550	12.961 481	11.456 439
1; 4	3; 3	-0.916 687	3.009 507	-3.883 9466	-3.792 147	-3.794 733	-3.354 102
3; 4	3; 3	-99.882 581	-127.061 082	-127.590 584	-127.588 524	-127.589 968	-127.787 323
1; 4	1; 4	-97.415 026	-108.055 071	-107.316 055	-107.334 482	-107.331 699	-105.038 286
3; 4	1; 4	-38.133 331	0	10.407 601	10.277 606	10.297 396	9.101 698
3; 4	3; 4	28.612 663	39.252 708	38.513 682	38.532 119	38.529 328	38.529 328
1; 5	1; 3	85.179 826	126.274 052	128.974 225	128.958 781	129.336 770	129.037 785
1; 5	3; 3	-0.269 230	-3.674 329	1.317 411	1.274 925	1.264 911	1.118 034
3; 5	1; 3	60.505 815	26.583 826	10.495 438	10.647 742	10.583 005	9.354 143
3; 5	3; 3	65.665 141	69.189 233	69.022 209	69.009 663	69.229 088	69.558 608
5; 5	3; 3	-76.363 826	16.305 010	14.425 467	14.515 278	14.491 377	16.201 852
1; 5	1; 4	-83.593 639	-69.460 806	-67.269 069	-67.283 818	-67.278 526	-65.453 419
1; 5	3; 4	82.77925	9.338 480	-5.939 770	-5.807 912	-5.796 551	-5.123 475 1
3; 5	1; 4	-16.878 049	13.723 799	14.185 688	14.212 501	14.198 591	12.549 900
3; 5	3; 4	-68.306 769	-135.235 541	-136.270 512	-136.278 511	-136.280 593	-136.610 395
5; 5	3; 4	68.027 894	5.913 498	9.628 153	9.450 810	9.486 833	10.606 602
1; 5	1; 5	-48.716 838	-95.531 881	-93.682 063	-93.716 833	-93.712 654	-96.231 811
3; 5	3; 5	43.126 580	-10.404 218	-12.110 821	-12.089 411	-12.091 955	-12.091 955
5; 5	5; 5	81.164 960	181.510 801	181.367 616	181.380 946	181.379 330	181.379 330
3; 5	1; 5	-52.087 438	0	12.414 743	12.299 948	12.313 845	10.884 004
5; 5	3; 5	113.765 385	0	5.258 647	5.006 770	5.038 315	5.633 007

These asymptotic expressions have been shown analytically to be accurate to order  $O((L/\Lambda)^4)$ . Their predictions are compared with numerically precise matrix elements below, in tables 1 and 2, and found, even for moderately large values of  $L/\Lambda$ , to be remarkably accurate; e.g.,  $\lesssim 0.2\%$  error for the states of  $L < 5$  of a  $(\lambda, \mu) = (32, 5)$  irrep. They are similar in form to those of an irrep of the  $rot(3)$  rigid-rotor algebra, given by

$$\langle KL' \| Q \| KL \rangle = \sqrt{2L+1} [(LK, 20|L'K) \bar{q}_0 + \delta_{K,1} (-1)^{\lambda+L+1} (L, -1, 22|L'1) \bar{q}_2], \quad (16)$$

$$\begin{aligned} \langle K+2, L' \| Q \| KL \rangle &= (-1)^{L'-L} \langle KL \| Q \| K+2, L' \rangle \\ &= \sqrt{(2L+1)(1+\delta_{K,0})} (LK, 22|L', K+2) \bar{q}_2, \end{aligned} \quad (17)$$

with

$$\bar{q}_0 = 2\lambda + \mu + 3, \quad \bar{q}_2 = \sqrt{\frac{3}{2}} \mu. \quad (18)$$

**Table 2.** Comparisons of quadrupole reduced matrix elements  $\langle K_f L_f \| \mathcal{Q} \| K_i L_i \rangle$  as described in table 1 for the  $SU(3)$  irrep (10, 4).

$K_i L_i$	$K_f L_f$	$\mathcal{Q}^{(1)} \cdot \mathcal{Q}^{(2)}$	I	II	III	A.S.	ROT(3)
0; 2	0; 0	19.146 198	25.227 104	26.854 801	26.823 096	26.832 816	27
2; 2	0; 0	20.625 775	12.473 680	8.415 442	8.515 925	8.485 281	6.928 203
0; 2	0; 2	-25.280 167	-33.827 282	-32.203 990	-32.280 594	-32.271 172	-32.271 172
2; 2	0; 2	-22.476 614	0	10.353 197	10.111 788	10.141 851	8.280 787
2; 3	0; 2	32.612 214	19.722 620	13.305 980	13.464 860	13.416 407	10.954 451
2; 3	2; 2	-30.272 798	-39.887 555	-42.461 171	-42.411 040	-42.426 407	-42.690 748
0; 4	2; 3	20.939 094	10.108 085	-9.182 851	-8.552 679	-8.485 281	-6.928 203
2; 4	2; 3	-32.438 270	-41.477 548	-41.183 272	-41.375 935	-41.366 653	-41.828 220
4; 4	2; 3	-18.966 068	5.276 214	8.367 394	8.079 759	8.197 561	8.197 561
0; 4	0; 2	29.639 536	34.848 900	41.886 357	41.742 972	41.815 923	43.296 321
0; 4	2; 2	0.058 923	-3.976 255	2.594 292	2.343 090	2.267 787	1.851 640
2; 4	0; 2	19.653 177	22.023 088	8.545 059	9.080 523	8.783 101	7.171 372
2; 4	2; 2	21.468 704	28.265 009	26.965 058	26.958 674	26.992 062	27.947 655
4; 4	2; 2	30.379 909	12.821 131	10.867 348	11.055 995	10.954 451	10.954 451
0; 4	0; 4	-45.391 345	-48.446 517	-40.877 618	-41.340 626	-41.281 422	-41.281 423
2; 4	2; 4	11.845 141	-9.673 158	-16.863 115	-16.470 046	-16.512 569	-16.512 569
4; 4	4; 4	33.546 210	62.755 119	57.740 733	57.810 677	57.793 992	57.793 992
2; 4	0; 4	-11.594 159	0	15.396 745	15.025 873	15.073 844	12.307 742
4; 4	2; 4	-33.333 631	0	5.247 793	4.721 849	4.854 239	4.854 239

The latter expressions give accurate approximations to the  $SU(3)$  matrix elements when both  $\lambda$  and  $\mu$  are large but are generally not as accurate as those given by the asymptotic approximations of equations (11)–(15).

A computationally simple method [9], used in the present calculations, for deriving numerically precise matrix elements of an  $su(3)$  irrep is to start from two known irreps,  $(\lambda_1, 0)$  and  $(\lambda_2, 0)$ , and diagonalize the  $SO(3)$ -invariant operator  $\mathcal{Q} \cdot \mathcal{Q}$  in the tensor product of these irreps, where  $\mathcal{Q} := \mathcal{Q}^{(1)} + \mathcal{Q}^{(2)}$  is the summed quadrupole tensor for the two irreps. To within a term proportional to the  $SO(3)$  Casimir invariant,  $\mathbf{L} \cdot \mathbf{L}$ , the operator  $\mathcal{Q} \cdot \mathcal{Q}$  is proportional to the  $SU(3)$  Casimir invariant. Thus, its eigenstates belong to  $SU(3)$  irreps and, in the process of deriving them, one obtains all the reduced matrix elements of the quadrupole tensor (albeit in a basis chosen arbitrarily by the computer). However, as shown in [9], if one then diagonalizes the operator  $\mathcal{Q}^{(1)} \cdot \mathcal{Q}^{(2)}$  within an irreducible  $(\lambda, \mu)$  subspace of the  $(\lambda_1, 0) \otimes (\lambda_2, 0)$  tensor product, then the degeneracies are lifted and the multiplicity of  $SO(3)$  irreps is resolved. Simple techniques for constructing such basis states and deriving their matrix elements were given in [9] and are used in the present calculations. Examples of reduced quadrupole matrix elements obtained in this way for the (32, 5) and (10, 4) irreps are shown in the columns of tables 1 and 2 labeled  $\mathcal{Q}^{(1)} \cdot \mathcal{Q}^{(2)}$ . However, this basis does not appear to correspond to any of the canonical bases we consider below.

### 3. Alternatives for resolving the $SO(3)$ multiplicities

We consider three alternatives.

#### 3.1. Alternative I

A standard way to resolve the  $SU(3) \supset SO(3)$  multiplicity is by the eigenstates of the angular-momentum-zero-coupled operator

$$X_3 := (\mathbf{L} \otimes \mathcal{Q} \otimes \mathbf{L})_0. \quad (19)$$

This operator is an  $SO(3)$  scalar in the  $SU(3)$  universal enveloping algebra [10]. Its potential use for resolving the  $SU(3) \supset SO(3)$  multiplicity was noted by Bargmann and Moshinsky [11]. Such a use is easily implemented because matrix elements of  $X_3$  in any  $SO(3)$ -coupled basis for an  $SU(3)$  irrep are given to within an unimportant  $L$ -dependent constant,  $c_L$ , by

$$\langle \beta L' \| X_3 \| \alpha L \rangle = \delta_{L',L} c_L \langle \beta L \| \mathcal{Q} \| \alpha L \rangle. \quad (20)$$

Thus, an  $SO(3)$ -coupled basis that diagonalizes  $X_3$  is given by the eigenstates of the  $M^L$  matrices with elements

$$M_{\beta\alpha}^L := \langle \beta L \| \mathcal{Q} \| \alpha L \rangle. \quad (21)$$

A variant of this method was used in the construction of bases for VCS irreps by  $K$ -matrix methods [12, 13]. Examples of reduced quadrupole matrix elements in such a basis are given in tables 1 and 2.

### 3.2. Alternative II

A second alternative is to use generally accepted  $SU(3)$  Clebsch–Gordan coefficients in an  $SO(3)$ -coupled basis to derive reduced matrix elements of the  $SU(3)$  quadrupole operator by means of the identity

$$\langle \beta L' \| \mathcal{Q} \| \alpha L \rangle = \left[ \frac{4}{3} (2L' + 1) (\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu) \right]^{\frac{1}{2}} ((\lambda\mu)\alpha L; (11)2 \| (\lambda\mu)\beta L'), \quad (22)$$

where  $(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu)$  is proportional to the value of the  $SU(3)$  Casimir operator for the  $(\lambda, \mu)$  irrep and  $((\lambda\mu)\alpha L; (11)2 \| (\lambda\mu)\beta L')$  is an  $SO(3)$ -reduced  $SU(3)$  Clebsch–Gordan coefficient.

In principle, the resolution of the  $SU(3) \supset SO(3)$  multiplicity, defined in this way, is only canonical to the extent that the Clebsch–Gordan coefficients are themselves expressed relative to a canonical basis. But, even if they are not, provided they are freely available, they serve the practical purpose of making it possible to compare the results of calculations by different researchers who use a common set of such coefficients. For present purposes, we use the Clebsch–Gordan coefficients of [7, 8]. Some results are shown for comparison with the other alternatives in tables 1 and 2. The comparisons show a remarkable similarity between these results and those of the following alternative. This will be explained in section 4.

### 3.3. Alternative III

A third alternative is given by basis states which diagonalize a specified linear combination of the  $SO(3)$  scalar operators  $X_3$  and

$$X_4 := (\mathbf{L} \otimes [\mathcal{Q} \otimes \mathcal{Q}]_2 \otimes \mathbf{L})_0 \quad (23)$$

within the space of an  $SU(3)$  irrep.

The rationale for choosing a particular linear combination is based on the observation that there is a natural resolution of the  $SO(3) \subset SU(3)$  multiplicity in the contraction limit in which an irrep of the  $su(3)$  algebra progresses asymptotically toward an irrep of the rotor-model algebra, denoted by  $rot(3)$ . In particular, as pointed out in [14], the intrinsic quadrupole moments of a  $rot(3)$  irrep, for which there is a naturally-defined  $SO(3)$ -coupled basis, are related to the  $SO(3)$  invariants  $\bar{X}_3$  and  $\bar{X}_4$ , where the latter operators are defined, as for the corresponding  $su(3)$  operators  $X_3$  and  $X_4$ , but in terms of the commuting  $rot(3)$  quadrupole operators. Because of the understood contraction of  $su(3) \rightarrow rot(3)$  for large values of  $\lambda$ ,

these observations suggested similar relationships for  $su(3)$ . Further relationships between the rigid-rotor model and the  $SU(3)$  model were developed by Leschber and Draayer [15].

The contraction of an  $su(3)$  irrep to an irrep of the  $rot(3)$  algebra is derived as follows [16]. Let

$$\epsilon(\lambda\mu) := \frac{1}{2}[\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu]^{-\frac{1}{2}} \quad (24)$$

denote the inverse square root of the eigenvalue of the  $SU(3)$  Casimir invariant

$$C_2 := Q \cdot Q + 3L \cdot L \quad (25)$$

for the irrep  $(\lambda, \mu)$ , and let  $Q$  denote the  $su(3)$  quadrupole tensor in inverse units of  $\epsilon(\lambda\mu)$ , i.e. the tensor with components

$$Q_\nu := \epsilon(\lambda\mu)Q_\nu. \quad (26)$$

It follows from equation (22) that, for values of  $L \ll \lambda$ , the non-zero reduced matrix elements of  $Q$  are of the order of magnitude

$$\langle \beta L' \| Q \| \alpha L \rangle \lesssim \left[ \frac{4}{3}(2L' + 1) \right]^{\frac{1}{2}}. \quad (27)$$

Moreover, the rhs of the commutation relation

$$[Q_\nu, Q_\mu] = 3\sqrt{10}(2\mu, 2\nu | 1\mu + \nu)\epsilon(\lambda\mu)^2 L_{\mu+\nu}, \quad (28)$$

becomes negligible when used with states of angular momentum  $L$  for which

$$\epsilon(\lambda\mu)^2 L \ll 1. \quad (29)$$

Thus, within the subspace of states of angular momentum  $L$  for which equation (29) is satisfied, the matrix elements of an  $su(3)$  irrep become indistinguishable from those of a  $rot(3)$  irrep. In this situation,  $su(3)$  is said to contract to  $rot(3)$ .

This contraction is of special interest in nuclear physics for explaining the origins of rotational structure in terms of the nuclear shell model in an  $SU(3) \supset SO(3)$  coupled basis. Because there is a natural resolution of the  $SO(3) \subset ROT(3)$  multiplicity, the  $su(3) \rightarrow rot(3)$  contraction suggests a parallel resolution of the  $SO(3) \subset SU(3)$  multiplicity.

It was suggested in [14] that the above-defined basis states of the  $rot(3)$  algebra should diagonalize a combination of the  $rot(3)$  operators,  $\bar{X}_3$  and  $\bar{X}_4$ , defined as for their  $su(3)$  counterparts but with commuting quadrupole operators. Thus, we considered the ratios of the matrix elements

$$R(L, K) := \frac{\langle K + 2, L \| \bar{X}_4 \| KL \rangle}{\langle K + 2, L \| \bar{X}_3 \| KL \rangle} = \frac{\langle K + 2, L \| [Q \otimes Q]_2 \| KL \rangle}{\langle K + 2, L \| Q \| KL \rangle} \quad (30)$$

for  $rot(3)$  irreps. Equations (16) and (17) give the reduced  $rot(3)$  matrix elements  $\langle KL \| Q \| KL \rangle$  and  $\langle K + 2, L \| Q \| KL \rangle$ . From them, we can evaluate

$$\langle K + 2, L \| [Q \otimes Q]_2 \| KL \rangle = \sum_{K_1 L_1} U(L2L2; L_1 2) \frac{\langle K + 2, L \| Q \| K_1 L_1 \rangle \langle K_1 L_1 \| Q \| KL \rangle}{\sqrt{2L_1 + 1}} \quad (31)$$

and the ratio  $R(KL)$  for any values of  $L$  and  $K$ . In this way, we determined the remarkable result that  $R(LK)$  takes the  $L$ - and  $K$ -independent value

$$R(LK) = \sqrt{\frac{8}{7}}\bar{q}_0. \quad (32)$$

This result means that the basis states of the rigid-rotor  $rot(3)$  algebra with good  $K$  quantum numbers are the eigenstates of the  $SO(3)$ -invariant

$$\bar{Z} := \bar{X}_4 - \sqrt{\frac{8}{7}}\bar{q}_0\bar{X}_3. \quad (33)$$



Similarly, we can define basis states for an  $SU(3)$  irrep to be the eigenstates of the corresponding  $SO(3)$ -invariant

$$Z := X_4 - \sqrt{\frac{8}{7}}(2\lambda + \mu + 3)X_3, \quad (34)$$

with the expectation that, in such a basis, the  $su(3)$  quadrupole matrix elements between states of  $L \ll \lambda$  will approach those of a  $rot(3)$  irrep in the asymptotic limit. Such basis states are uniquely defined and provide a physically relevant resolution of the  $SO(3)$  multiplicity for any  $SU(3)$  irrep. Results obtained for such  $SU(3)$  bases are shown in tables 1 and 2.

#### 4. Discussion

Tables 1 and 2 show comparisons of reduced quadrupole matrix elements obtained for the alternatives given above for defining orthonormal  $SO(3)$ -coupled basis states for  $SU(3)$  irreps. The tables also show the corresponding results given by the asymptotic approximation of equations (11)–(15) and for the  $rot(3)$  matrix elements given by equations (16) and (17). It should be emphasized that the results given for the  $SU(3)$  matrix elements listed in the columns headed I, II and III are all numerically accurate to the precision shown; they only differ to the extent that they were computed relative to different bases. The asymptotic results in the column headed A.S. are expected to agree with those of column III for the values of  $2\lambda + \mu \gg L_i$ . Those listed in the column headed  $ROT(3)$  are for the rotor algebra and likewise are expected to approximate those of columns III and A.S. when both  $\lambda \gg L_i$  and  $\mu \gg L_i$ .

It can be seen that the matrix elements of alternatives II and III differ very little and both are close to those of the asymptotic approximation, A.S. In fact, the rms differences between the entries of any of these three columns in table 1 are of the order 0.015. The near equivalence of the II and III results is remarkable and fortuitous in the sense that it implies that the bases used in the calculation of  $SU(3) \supset SO(3)$  Clebsch–Gordan coefficients in [7, 8] are, in fact, near canonical in the above-defined sense of the basis states being the eigenstates of a Hermitian operator. This result was unexpected because the choice of basis states for an  $SU(3)$  irrep used in the computation of the  $SU(3)$  Clebsch–Gordan coefficients given in [7, 8] did not make use of the  $SO(3)$ -invariant operator,  $Z$ . However, the construction of the basis states that was used did make use of rotor-model methods which, in the asymptotic limit, likewise give standard  $rot(3)$  results. Thus, in retrospect, it is understood that the Clebsch–Gordan coefficients obtained should be consistent with the  $SU(3)$  bases states defined by alternative III. Note that the results in columns II and III are precise and differ only because they are calculated relative to slightly different bases. Thus, the fact that they are so close to the asymptotic results is a direct indicator of the accuracy of the latter for finite-dimensional irreps (they are, by construction, precise in the asymptotic limit). The results of the A.S. asymptotic approximation are closest to those of alternative III; they are closer (generally much closer) than 1% for the matrix elements shown in table 1 and closer (generally much closer) than 3% for those shown in table 2.

The results of alternative I, obtained by diagonalizing the  $SO(3)$  invariant,  $X_3$ , are qualitatively similar but much less close to those of the asymptotic rotor-model limit than those obtained by diagonalizing the linear combination of  $X_3$  and  $X_4$ , given by  $Z$  in equation (34).

The tabulated matrix elements also show the expected result that the accurate matrix elements for the basis III are reproduced much more accurately, for small values of  $\mu$  by the asymptotic  $SU(3)$  results of column A.S. than by those of the  $ROT(3)$  limit.

In conclusion, we remark that the above results provide a physical and practical resolution of the so-called inner, i.e.,  $SU(3) \supset SO(3)$ , multiplicity problem. However, the outer

multiplicity that occurs in the decomposition of tensor products of  $SU(3)$  irreps is also of importance and, at present, we know of no canonical way to resolve it.

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